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1997 J. Phys. A: Math. Gen. 30 L643

(http://iopscience.iop.org/0305-4470/30/18/006)

J. Phys. A: Math. Gen. **30** (1997) L643–L649. Printed in the UK PII: S0305-4470(97)84754-3

LETTER TO THE EDITOR

Damage spreading in random field systems

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Received 3 June 1997

Abstract. We investigate how a quenched random field influences the damage-spreading transition in kinetic Ising models. To this end we generalize a recent master equation approach and derive an effective field theory for damage spreading in random-field systems. This theory is applied to the Glauber Ising model with a bimodal random-field distribution. We find that the random field influences the spreading transition by two different mechanisms with opposite effects. First, the random field favours the same particular direction of the spin variable at each site in both systems which reduces the damage. Second, the random field suppresses the magnetization which in turn tends to increase the damage. The competition between these two effects leads to a rich behaviour.

The central question of damage spreading (DS) [1–3] is how a small perturbation in a cooperative system changes during the time evolution. This is analogous to the question to what extent the time evolution depends on the initial conditions, one of the main questions in non-linear dynamics that lead to the discovery of chaotic behaviour [4]. In order to study DS the simultaneous time evolution of two replicas of a cooperative system is considered. The two replicas evolve stochastically under the same noise realization (i.e. the same random numbers are used in a Monte Carlo procedure). The differences in the microscopic configurations of the two replicas are then used to characterize the dynamics and to distinguish regular and chaotic phases, depending on external parameters.

Among the simplest cooperative systems are kinetic Ising models where DS has been investigated quite intensively within the last years using different dynamical algorithms such as Glauber [3, 5–8] or heat-bath dynamics [2, 7, 9, 10]. In contrast to the equilibrium critical behaviour the results of DS do depend on the particular choice of the dynamical algorithm although recently an attempt has been made to give a more objective definition of DS [11]. In general, there are two different mechanisms by which damage can spread in a kinetic Ising model. First, the damage can spread *within* a single ergodic component (i.e. a pure state or free-energy valley) of the system. This is the case for Glauber or Metropolis dynamics. Second, the damage can spread when the system selects one of the free-energy valleys at random after a quench from high temperatures to below the equilibrium critical temperature. This is the only mechanism to produce DS in an Ising model with heat-bath dynamics. This algorithm is thus well suited for exploring the structure of the free-energy landscape.

In the literature the name DS has been applied not only to the studies discussed above but also to a different though related type of investigation in which the two systems are *not* identical. Instead, one or several spins in one of the copies are permanently fixed

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in one direction. Thus the equilibrium properties of the two replicas deviate from each other and their microscopic differences can be related to equilibrium correlation functions [12, 13]. Note that in these works the use of identical noise (i.e. random numbers) for the two systems is not essential but only a method to reduce the statistical error.

Whereas DS in clean Ising models is comparatively well understood less is known about disordered models. The influence of random fields has been investigated in a two-dimensional Ising-like model with Metropolis dynamics giving a reduction of the damage at high temperatures but an increase at low temperatures [14]. By using the heat-bath algorithm DS has been used to study the phase-space structure of Ising spin glasses [15–17] and the corresponding critical behaviour at the DS transition [18].

In this letter we consider the original DS problem, namely the time evolution of two identical systems and study the influence of a quenched random field on DS in kinetic Ising models. To this end we generalize the master-equation approach [7, 8] to random-field systems. The resulting effective-field theory of DS is then applied to the Glauber Ising model with a bimodal random-field distribution. We study the dependence of the spreading transition on temperature and field and determine the phase diagram.

We consider two identical Ising models with N sites described by the Hamiltonians $H^{(1)}$ and $H^{(2)}$ given by

$$H^{(n)} = -\frac{1}{2}\sum_{ij} J_{ij} S_i^{(n)} S_j^{(n)} - \sum_i \varphi_i S_i^{(n)} \tag{1}$$

where $S_i^{(n)}$ is an Ising variable with the values ± 1 and $n = 1, 2$ distinguishes the two replicas. J_{ij} is the (non-random) exchange interaction between the spins. The random-field values φ_i are chosen independently from a distribution $\rho(\varphi)$. The dynamics of the systems are given by stochastic maps $S_i^{(n)}(t+1) = F[\{S_j^{(n)}(t)\}]$, e.g. the Glauber algorithm

$$S_i^{(n)}(t+1) = \mathrm{sgn}[v[h_i^{(n)}(t)] - \tfrac{1}{2} + S_i^{(n)}(t)(\xi_i(t) - \tfrac{1}{2})] \tag{2}$$

where the transition probability $v(x)$ is given by the usual Glauber expression

$$v(x) = \mathrm{e}^{x/T}/(\mathrm{e}^{x/T} + \mathrm{e}^{-x/T}). \tag{3}$$

Here $h_i^{(n)}(t) = \sum_j J_{ij} S_j^{(n)}(t) + \varphi_i$ is the local magnetic field at site i and (discretized) time t in the system n. $\xi_i(t) \in [0, 1)$ is a random number which is identical for both systems, and T denotes the temperature.

Within the master equation approach [7, 8] the simultaneous time evolution of the two replicas is described by the probability distribution

$$P(\nu_1, \ldots, \nu_N, t) = \left\langle \sum_{\nu_i(t)} \prod_i \delta_{\nu_i, \nu_i(t)} \right\rangle \tag{4}$$

where $\langle\cdot\rangle$ denotes the average over the noise realizations. The variable ν_i with the values $++$, $+-$, $-+$, or $--$ describes the states of the spin pair $(S_i^{(1)}, S_i^{(2)})$. The distribution P fulfils the master equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} P(\nu_1, \ldots, \nu_N, t) = &-\sum_{i=1}^{N} \sum_{\mu_i \neq \nu_i} P(\nu_1, \ldots, \nu_i, \ldots, \nu_N, t) w(\nu_i \to \mu_i) \\ &+ \sum_{i=1}^{N} \sum_{\mu_i \neq \nu_i} P(\nu_1, \ldots, \mu_i, \ldots, \nu_N, t) w(\mu_i \to \nu_i).\end{aligned} \tag{5}$$

The transition probabilities $w(\mu_i \to \nu_i)$ have to be calculated from the properties of the stochastic map F which defines the dynamics.

As in the clean case we derive an effective-field theory by assuming that fluctuations at different sites are statistically independent which amounts to approximating the distribution $P(\nu_1, \ldots, \nu_N, t)$ by a product of single-site distributions $P_{\nu_i}(i, t)$. However, in a disordered system different sites are not equivalent and thus their $P_{\nu_i}(i, t)$ are not identical. This is the main difference to the clean case [7, 8] where the single-site distributions are all identical. In the following we further assume that $P_{\nu_i}(i, t)$ is determined by the local value φ_i of the random-field only, $P_{\nu_i}(i, t) \equiv P_{\nu_i}(\varphi_i, t)$. In general, this is an approximation since sites with identical random-field values may well have different environments which should influence the distribution. In the mean-field limit of infinite dimensions over infinite-range interactions, however, the above assumption becomes exact.

Inserting the decomposition

$$P(\nu_1, \ldots, \nu_N, t) = \prod_{i=1}^{N} P_{\nu_i}(\varphi_i, t) \tag{6}$$

into the master equation (5) gives a system of coupled equations of motion for the single-site distributions

$$\frac{\mathrm{d}}{\mathrm{d}t} P_\nu(\varphi) = \sum_{\mu \neq \nu} [-P_\nu(\varphi) W(\nu \to \mu, \varphi) + P_\mu(\varphi) W(\mu \to \nu, \varphi)] \tag{7}$$

where $W(\mu \to \nu, \varphi)$ is the transition probability w averaged over the states ν_i of all sites. We now define the local damage $d(\varphi) = P_{+-}(\varphi) + P_{-+}(\varphi)$. The total damage is obtained as the corresponding disorder average

$$D = \left\langle \frac{1}{2N} \sum_{i=1}^{N} |S_i^{(1)} - S_i^{(2)}| \right\rangle = \int \mathrm{d}\varphi\, \rho(\varphi)\, \mathrm{d}(\varphi). \tag{8}$$

The equation of motion of the local damage can be easily determined from (7). Using some symmetry relations [8] between the transition probabilities W, it reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} d(\varphi) &= [1 - d(\varphi)][W(-- \to +-, \varphi) + W(-- \to -+, \varphi)] \\ &+ d(\varphi)[-1 + W(-+ \to +-, \varphi) + W(-+ \to -+, \varphi)] \end{aligned} \tag{9}$$

So far the considerations have been rather general, being valid for any (single-site) dynamic rule and any distribution of the random field. We now apply the formalism to the Glauber Ising model in the mean-field limit $J_{ij} = J_0/N$ (for all i and j). The random-field distribution remains unspecified so far. In order to determine the spreading point for infinitesimal initial damage it is sufficient to solve (9) in linear order in $d(\varphi)$. After calculating the transition probabilities W in analogy to the clean case [8] and some further algebra we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} d(\varphi) = -|m(\varphi)| d(\varphi) + \frac{J_0}{T}[1 - m^2(\varphi)] D. \tag{10}$$

If we concentrate on DS processes starting in equilibrium conditions the local magnetization

$$m(\varphi) = \tanh[(J_0 m + \varphi)/T] \tag{11}$$

and the average magnetization m are time-independent. Equation (10) is very similar to the corresponding equation (36) of [8] for DS in a homogeneous field. The main difference is that for random-field systems we have to distinguish between the local damage $d(\varphi)$ which determines the healing probability (first term in (10)) and the average damage D which determines the damaging probability of a site (second term in (10)). In a homogeneous

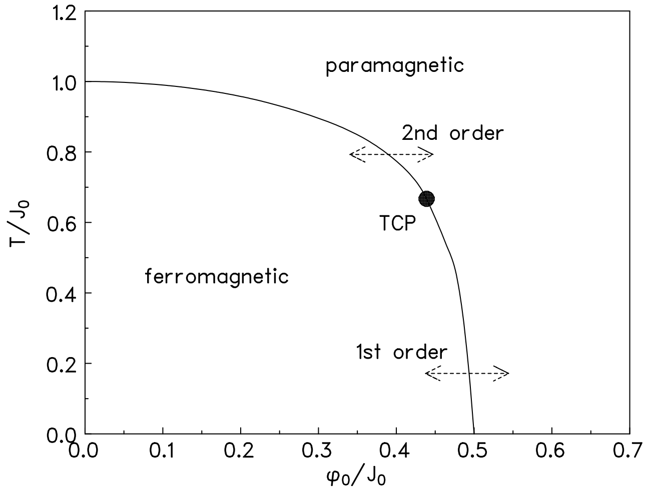

Figure 1. Thermodynamic phase diagram of the mean-field Ising model with bimodal random field. TCP denotes the tricritical point.

system local and average damage are identical. Consequently, replacing $d(\varphi)$ by D and $m(\varphi)$ by m in (10) exactly gives the corresponding equation for the homogeneous system.

To proceed further we have to specify the random-field distribution $\rho(\varphi)$. As an example we will discuss the bimodal distribution

$$\rho(\varphi) = \tfrac{1}{2}[\delta(\varphi - \varphi_0) + \delta(\varphi + \varphi_0)] \qquad (\varphi_0 > 0). \tag{12}$$

The thermodynamics of the mean-field Ising model with a bimodal random field was investigated in detail almost 20 years ago [19]. The equation of state takes the form

$$m = \frac{1}{2}\left[\tanh\left(\frac{J_0 m + \varphi_0}{T}\right) + \tanh\left(\frac{J_0 m - \varphi_0}{T}\right)\right]. \tag{13}$$

The resulting phase diagram is summarized in figure 1. There is a tricritical point at $T_{\mathrm{TCP}} = 2J_0/3$ and $\varphi_{\mathrm{TCP}} \approx 0.439 J_0$. For $T > T_{\mathrm{TCP}}$ the ferromagnetic phase transition is of second order, for $T < T_{\mathrm{TCP}}$ it is of first order.

We now turn to our results on DS in this model. Using the notations $d_\pm = d(\pm\varphi_0)$, $\boldsymbol{d} = (d_+, d_-)$ and $m_\pm = m(\pm\varphi_0)$ the equation of motion (10) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{d} = \boldsymbol{A}\cdot\boldsymbol{d}. \tag{14}$$

The dynamical matrix is given by

$$\boldsymbol{A} = \begin{bmatrix} -|m_+| + (1 - m_+^2)\frac{J_0}{2T} & (1 - m_+^2)\frac{J_0}{2T} \\ (1 - m_-^2)\frac{J_0}{2T} & -|m_-| + (1 - m_-^2)\frac{J_0}{2T} \end{bmatrix}. \tag{15}$$

The question whether the damage spreads or heals can be answered by means of the eigenvalues of $\boldsymbol{A}$. If both eigenvalues are negative the damage heals, if at least one of them is positive the damage spreads.

In the paramagnetic phase we have $|m_+| = |m_-| = \tanh(\varphi_0/T)$. The eigenvalues of $\boldsymbol{A}$ are given by $\lambda_1 = -m_+ + (1 - m_+^2)J_0/T$ and $\lambda_2 = -m_+$. The corresponding eigenmodes are the average damage D and the damage difference $d_+ - d_-$, respectively. Consequently,

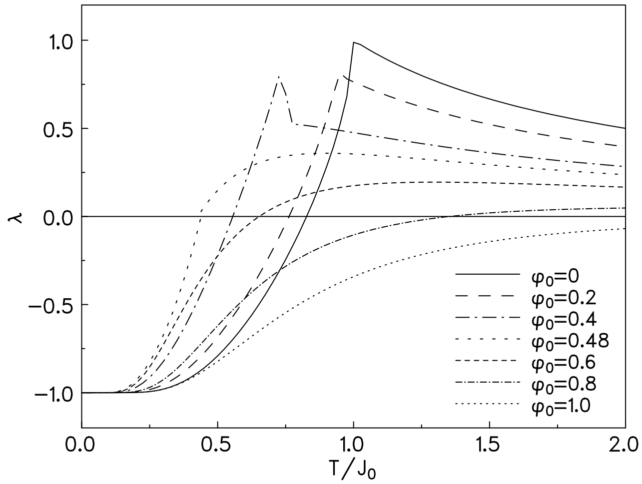

Figure 2. Lyapunov exponents of the mean- field Glauber Ising model with bimodal random field. The peak in the curve for $\varphi_0 = 0.4$ corresponds to the T where m_- vanishes.

in the paramagnetic phase the Lyapunov exponent which is given by the largest eigenvalue of $\boldsymbol{A}$ reads

$$\lambda = -\tanh(\varphi_0/T) + [1 - \tanh^2(\varphi_0/T)]J_0/T. \tag{16}$$

Its dependence on temperature and random-field strength is visualized in figure 2. In the paramagnetic phase the Lyapunov exponent decreases with increasing random field φ_0 and therefore the spreading temperature T_s which is defined by $\lambda = 0$ increases. This can be easily understood from the fact that a random field favours a particular spin direction at each site. Since this direction is the same for the two replicas the corresponding spins in the two replicas tend to be parallel which reduces the damage. For random-field strength $\varphi_0 > J_0$ the Lyapunov exponent remains negative for all temperatures and thus the damage never spreads. Asymptotically for $\varphi_0 \to J_0$ we obtain

$$T_s^2 = \frac{2}{3} J_0^2 \frac{1}{1 - \varphi_0/J_0}. \tag{17}$$

We note that the critical φ_0 which completely suppresses DS has the same value as the corresponding critical homogeneous field [8] although the functional dependence of T_s on the field is different.

In the ferromagnetic phase $|m_+|$ and $|m_-|$ are different. In this case the eigenvalues of $\boldsymbol{A}$ are given by

$$\lambda_{1,2} = \frac{1}{2}\left[-|m_+| - |m_-| + \frac{J_0}{2T}(1 - m_+^2 + 1 - m_-^2)\right] \pm \left[\frac{1}{4}(|m_+| - |m_-|)^2 + \frac{J_0^2}{16T^2}(1 - m_+^2 + 1 - m_-^2)^2 + \frac{J_0}{4T}(|m_+| - |m_-|)(m_+^2 - m_-^2)\right]^{1/2}. \tag{18}$$

In order to calculate the Lyapunov exponent we first determine the average magnetization m as a function of φ_0 and T from the equation of state (13). We then calculate m_+ and m_- and insert them into (18). The resulting Lyapunov exponents are presented in figure 2. In contrast to the paramagnetic phase the spreading temperature *decreases* with increasing random-field strength. At a first glance this seems to contradict the argument given above,

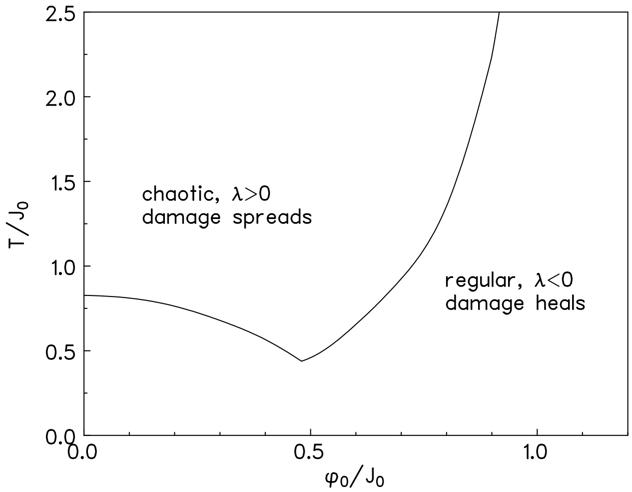

Figure 3. Damage-spreading phase diagram of the mean-field Glauber Ising model with a bimodal random field.

namely that a random field favours a particular spin direction and thus reduces the damage. However, the random field also influences DS via a reduction of the magnetization since the Lyapunov exponent (18) is determined by the local magnetizations. In the ferromagnetic phase this effect is stronger than that of the preferred orientation discussed above and thus T_s is reduced.

By means of (16) and (18) we have determined the spreading temperature as a function of the random-field strength. The resulting phase diagram of DS in the mean-field Glauber Ising model with bimodal random field is shown in figure 3. The minimum spreading temperature $T_{s,\min} \approx 0.438$ is obtained when the spreading transition coincides with the ferromagnetic phase transition which occurs for $\varphi_0 \approx 0.480$ (see figure 2).

To summarize, we investigated the influence of a quenched random field on DS in kinetic Ising models. We generalized the master-equation approach [7, 8] to random-field systems and derived an effective field theory for DS. As an example we studied the mean-field Glauber Ising model with bimodal random field. We found that the random field supports the spreading of damage in the ferromagnetic phase but hinders it in the paramagnetic phase. For strong enough field the damage never spreads.

In the concluding paragraph we discuss other random-field distributions and compare our results with the numerical simulation [14]. The influence of the particular form of the random-field distribution on DS can be discussed qualitatively by means of (10). This equation shows that the healing probability is proportional to the local magnetization. This means that the damage on sites with local magnetization zero cannot heal. Consequently, DS will be qualitatively different in systems with a continuous random-field distribution since even for very strong random fields there will be sites with vanishing local magnetization. Thus damage will spread on a subset of sites with low enough random field. However, with $T \to 0$ the measure of this subset goes to zero. A detailed investigation of this case will be published elsewhere [20]. These results also help to understand the numerical simulation [14] which was carried out for a box distribution. It shows a decrease of the spreading temperature with increasing random field although the stationary value of D is reduced at high temperatures. This is consistent with a reduction of T_s due to a suppression of the local magnetization and spreading on a subset of sites at low temperatures. However, a direct comparison with the mean-field theory is not possible since in the simulations a two-dimensional system was used which—due to fluctuations—does not have an ordered phase for any finite random field. Finally, we discuss possible extensions of this work. Besides a systematic investigation of different random-field distributions the damage equation of motion should be solved beyond first order in the damage. This will permit the determination of the stationary damage values and the investigation of the critical behaviour at the spreading transition. Some studies along these lines are in progress [20].

This work was supported in part by the DFG under grant numbers Vo659/1-1 and SFB393 and by the NSF under grant number DMR-95-10185.

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